MODELLING FORMING PROCESSES

(study aid)

Doc. Ing. Richard Fabík, Ph.D.

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PRINCIPALS OF MATHEMATICAL MODELLING.

Objective: After studying his chapter through you will be able to:

- Clarify basic concepts from the area of mathematical modelling
- To design inputs and outputs while controlling forming processes
- To compare the possibilities of physical and mathematical modelling
- To explain the basic procedure during mathematical modelling
- To determine unknown quantities and peripheral conditions for the simplest tasks
- To derive a model for a 1D elasticity task

Content of Chapter 1

1.1 Mathematical modelling in industrial research and development

1.2 Basic principles of mathematical modelling

1.1 Mathematical modelling in industrial research and development

Study time: 0,5 hours

Concepts to remember

CAD, CAE, CAM, model, limiting conditions, optimization

Lecture
Obr. 1.1.  Block diagram for process design and control in metal forming [01]

(The terms for this diagram are on the last page in English)

∑ Summary

Only a thorough knowledge of all variables into input processes and their effects on the forming process enable us to carry out its optimization. The development of specialized IT programs helps us very much.
1.2 Basic principles of mathematical modelling

**Study time:** 1 hour

**Concepts to remember**
- CAD, CAE, CAM, model, limiting conditions, optimalization

**Lecture**

- **Introduction**

  There exists two basic methods of modelling: experimental modelling, which is not always possible to implement and theoretical modelling, which usually requires a certain simplification of assumptions. It is possible to carry out theoretical modelling by two methods: analytically (the solution is exact, but it is attainable only for the simplest tasks) or numerically (approximately) using IT (this method is indicated as mathematical modelling).

  The basic procedure in mathematical modelling is shown in the following diagram in Fig 1.2. We can allow for mistakes in each of the indicated steps!

  ![](image)

  **Fig. 1.2. Basic procedure in mathematical modelling**
  
  Reality, object, phenomenon Mathematical model Solvable mathematical model Numerical methods and PC calculations Illustration solutions Making conclusions

- **Relations between the quantities of a usual mathematical model**

  A mathematical model usually is represented by a differential equation. It concerns defining important quantities and their mutual relations. It is generally possible to describe the common relations between quantities as in Fig. 1.3.

  ![](image)

  **Fig. 1.3. Relations between quantities of a general mathematical model**
Physical relation

where $u$ (unknown) is unknown (heat; displacement),

$\varepsilon = du/dx$ (temperature drop, deformation),

$\tau = f(\varepsilon)$ (heat flow, Fourier's Law, tension, Hook's Law for elastic deformation, or for example, Hollomon's for plastic deformation),

$f$ symbolizes an internal source (internal heat source, gravitational force) and $d\tau/dt = f$.

In general cases we then look for the solution by the Poisson's equation:

$$-\Delta u = f \quad \text{in} \quad \Omega,$$

with a homogeneous Dirichlet's peripheral condition

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where $\Omega$ is the polyhedronic area in $\mathbb{R}^d (d \geq 1)$ with boundary $\partial\Omega$, $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_d^2}$ and $f \in L^2(\Omega)$ is the entered function.

The analytic solution of a task is not generally known and so nothing remains for us than only to approximately solve for examples using the finite element method.

Let's now introduce two examples:
Example 1

The task of heat conduction in a one-dimensional space (1D)

We have a round bar with length $l$ and cross-section $a$. The bar surface with the exception of the cylinder base is insulated and doesn’t conduct any heat ($q = 0$).

Main quantities:

- temperature $T$ (constant in the cross-section)
- the density of heat flow $q$, the amount of heat for the cross-section content unit per time unit
- temperature drop $\varepsilon_{1,2} = (T_1 - T_2)/(x_1 - x_2)$

If it is $(x_1 \to x_2)$ then $\varepsilon_x = \lim_{l \to 0} \frac{T(x + l) - T(x)}{l} = T'(x)$
Example 1 - continuation

Furier’s Law of heat conduction (the density of heat flow is the proportional temperature gradient):

\[-\lambda \frac{\partial T}{\partial x} = q\]

Where \( \lambda \) is the coefficient of thermal conductivity [W.m\(^{-1}\).K\(^{-1}\)],
\( q \) is the density of heat flow [W.m\(^{-2}\)].

Internal temperature formation (heat arises for example deformation, phase transformation, passage of current and other areas.)

\[ Q(x_1, x_2) = \int_{x_1}^{x_2} a \cdot f(x)dx = \int_{x_1}^{x_2} a \cdot R \cdot I^2dx \]

Where:
\( a \) is the bar cross-section (m\(^2\))
\( R \) is electrical resistance [W]
\( I \) is electrical current [A]

Peripheral conditions
in our case we can have either a constant temperature or heat flow

\[ T(0) = \hat{T} \quad \text{or} \quad T(0) = \tilde{T} \]
\[ q(l) = \hat{q} \quad \text{or} \quad q(l) = \tilde{q} \]

Verbally: temperature \( T \) at point 0 is equal to some specific value \( \hat{T} \), the density of heat flow \( q \) going through point \( l \) is equal to specific value \( \hat{q} \).

Mathematical model (stationary model – quantities don’t change in time)

Basic principle: The amount of heat in any part of the bar is zero.

\[-q'(x) + Q = 0\]
\[-(\lambda \cdot T'(x))' = Q \quad \forall (0, l)\]

Non-linear model:
\[ \lambda = f(T) \]
Example 2

A 1D elastoplastic task
We have a round bar with length $l$ and cross-section $a$, which is solidly held at one end (for example, in the jaws of a drawing machine). The bar is loaded by force $F$ in the length direction.

Main quantities:
- displacement $u$, cross-sections are not deformed!
- tension $\sigma$, force acting on surface unit
- relative deformation $\varepsilon$

\[
\varepsilon_x = \lim_{h \to 0} \left[ \frac{h + u(x + h) - u(h)}{h} \right] - h = \lim_{h \to 0} \frac{u(x + h) - u(h)}{h} = u'(x)
\]

Hook’s Law (tension is direct proportional deformation):
\[
\sigma = E \cdot \varepsilon
\]
Where $E$ is the elasticity model at tension [MPa],

Internal source (gravitational force).
\[
f = \rho \cdot g
\]
where $f$ is gravitational force per volume unit [N.m$^{-3}$]
- $\rho$ is the density [kg.m$^{-3}$]
- $g$ gravitational acceleration [m.s$^{-2}$]

Marginal conditions
\[
u(0) = 0
\]
\[
\sigma(l) = \hat{\sigma}
\]

Verbally: displacement $u$ at point 0 is equal to 0 (the bar is solidly held by a machine’s jaws). On the bar end at point $l$ there acts the specific value of tension $\hat{\sigma}$.

Mathematical model (stationary model – quantities do not change in time)
\[
- \sigma'(x) + f = 0
\]
\[
-(E \cdot \varepsilon'(x))' = f \quad \forall \langle 0, l \rangle
\]
• Derivation of a stationary, mathematical 1D model of an elasticity task.

From the bar in Example 2 we set up an infinitely short element (viz. Fig. 1.4.). We now write some of the forces, which act on the given element.

\[ -\sigma \cdot S + (\sigma + d\sigma) \cdot S + f(x) \cdot dx \cdot S = 0 \]  \hspace{1cm} (1.3)

Since the element sticking out is infinitely small and its mass is close to zero. In calculating the affects of gravitation there must be used \( f(x) \) which is gravitational force per volume unit [N.m\(^{-3}\)].

By correction we get the form:

\[ -\sigma + \sigma + d\sigma + f(x) = 0 \]  \hspace{1cm} \hspace{1cm} (1.4)

and thus:

\[ \frac{d\sigma}{dx} = -f(x). \]  \hspace{1cm} \hspace{1cm} (1.5)

If we enter for \( \sigma \) from Hook’s Law \( \sigma = E \cdot \varepsilon \) we get this form:

\[ \frac{d}{dx} (E \cdot \varepsilon) = -f(x) \]  \hspace{1cm} (1.6)

We enter \( \varepsilon = \frac{du}{dx} \) and we get the final form:

\[ \frac{d}{dx} \left( E \cdot \frac{du}{dx} \right) = -f(x) \]  \hspace{1cm} (1.7)

\[ \text{neboli} \]  \hspace{1cm} (1.7)

\[ E \cdot u = -f(x) \]  \hspace{1cm} (1.7)
Example 3

Analytic solution of a stationary 1D elasticity task

We solve it using a differential equation of the second order (1.7)

We intergrate the equation:

\[ E \cdot \frac{du}{dx} = -\int \omega f(\omega) \cdot d\omega + C_1, \]

For determining constant \( C_1 \) we use a marginal condition:

\( \sigma(1) = P \) (the tension in point 1 is equal to \( P \))

In point 1 then it is valid that:

\[ E \cdot \left( \frac{du}{dx} \right)_{1} = -\int_{1} f(\omega) \cdot d\omega + C_1 \]

\[ P = C_1 \]

We install it into the equation and correct it:

\[ E \cdot \frac{du}{dx} = -\int_{1} f(\omega) \cdot d\omega + P \Rightarrow \frac{du}{dx} = -\frac{1}{E} \int_{1} f(\omega) \cdot d\omega + \frac{P}{E}, \]

We integrate for a second time:

\[ u = -\frac{1}{E} \cdot \int_{0}^{x} \int_{1}^{y} f(\omega) \cdot d\omega \cdot dv + \frac{P}{E} \cdot x + C_2, \]

\( u(0) = 0 \) (displacement at point 0 is zero)

At point 0 it is valid that:

\( u(0) = 0 + 0 + C_2 \)

Thus the final equation describing the displacement of any point on the bar depending on its position looks like this:

\[ u = -\frac{1}{E} \cdot \int_{0}^{x} \int_{1}^{y} f(\omega) \cdot d\omega \cdot dv + \frac{P}{E} \cdot x \]

Now let’s install for \( f(\omega) \) some specific value, we have two sensible possibilities:

1. If gravitation does not act the function will be: \( f(\omega) = 0 \)

And thus the sought function has the form of a straight line: \( u = \frac{P}{E} \cdot x \)

2. If gravitation acts the function will be: \( f(\omega) = \rho \cdot g = \frac{F_g}{V} = C \), kde \( F_g = m \cdot g \)

\[ u = -\frac{1}{E} \cdot \int_{0}^{x} \int_{1}^{y} C \cdot d\omega \cdot dv + \frac{P}{E} \cdot x \]

\[ u = -\frac{1}{E} \cdot \int_{0}^{x} \int_{1}^{y} C \cdot x \cdot dv + \frac{P}{E} \cdot x = -\frac{1}{E} \cdot \int_{0}^{x} (C \cdot v - C) d\omega + \frac{P}{E} \cdot x \]

\[ u = -\frac{1}{E} \cdot \left[ C \cdot v^2 - C \cdot x \right]_{0}^{x} + \frac{P}{E} \cdot x \]

And thus the sought function has the form of a parabola:

\[ u = -\frac{1}{E} \left( C \cdot \frac{x^2}{2} - C \cdot x \right) + \frac{P}{E} \cdot x \]
Summary

There is experimental and theoretical modelling, and each has its advantages and limitations. Theoretical modelling using numerical methods is called mathematical modelling. The procedure during mathematical modelling of some phenomena comes from a mathematical model (for example partial, differential equations), through a solvable mathematical model (system of equation on an unknown obtained for example by using the finite element method), there follows a numerical solution for computers and illustrating the solution. Each step can lead to a mistake, the task of the engineer is then to compare results and text phenomena to see if the results describe the reality.

Questions for Chapter 1

1.1. Can you present an example of a mathematical model and a solvable mathematical model for metal forming?

1.2. Do you know some numerical method for solving systems of linear equations?

1.3. What are peripheral conditions used for?

1.4. How is it possible to describe the relation between tension and deformation?

1.5. What is the difference between stationary and non-stationary tasks?

1.6. How is simplification used in Example 1 and is it possible to reach this simplification in practice?

1.7. How is simplification used in Example 2 and is it possible to reach this simplification in practice?

1.8. Can you define relative deformation using displacement?

Tasks to be solved in Chapter 1

1.1. Analytically solve the 1D task of heat conduction from Example 1?

1.2. Outline the results in Example 3 into the graph \( u = f(x) \) for both variants.

Other sources


2. FINITE ELEMENT METHOD

Objective: After studying this chapter through you will be able:
- To speak about the history of the final element method.
- To explain the basic principles of the finite element method.
- To describe the effect of element size for solving mathematical tasks
- To construct basic functions, and to derive their equations and derivations
- To set up a stiffness matrix for a 1D elasticity task.
- To explain the pitfalls of numerically solving linear equation systems.

The content of Chapter 2

2.1 Introduction
2.2 Finite element method in practice
2.3 Finite element method principle

2.1. Introduction

Study time: 0,5 hours

Concepts to remember

Finite element method (MKP), variation principles, triangulation, stiffness matrix

Lecture

The method was established in a period around the year 1956 at The research Institute of Aeronautic and Cosmic Mechanics in Ohio, USA (Wright Paterson Air Force Base). The research team was lead by Prof. R.W. Cloughem, and there especially participated R.L. Melosh, H.C. Martin, J.L. Tocher and others. The research and development of the mentioned numerical method brought about the strict requirements of the Apollo „moon“ program in the area of the development and in the design of load-bearing rockets. At a given time and with a known volume of finance (3 billion) and from analysis it was determined that using the experiment cannot fulfill the task. One method remained and that is the development of such a numerical method, which would be necessary for calculating a project of a new type of rocket and of other systems managing the Apollo project. The research results were further intensively used at stated military bases for the design of airplanes, submarines, and missiles.
An interesting thing

Engineers using long-term methods had successfully calculated them before mathematicians proved the convergence method, and especially gave light to those who spent decades calculating.

In the civil sector the most stormy final element method (FEM) developed in the years 1965-1975. The first promotor and an unwavering champion of the method in Czechoslovakia was Prof. V. Kolář, DrSc. from Brno, who also attained a well-known international recognition for the program of the NE Council. Using FEM today solves a whole number of tasks, whose implementation was not yet possible and it is not only in the areas of the mechanics of connected bodies or systems. Its own general mathematical formula enables FEM to solve these problems: rock mechanics, the flow of liquids and gases, the expansion of heat and radiation, stationary and non-stationary electromagnetic fields, etc. There are even known experiments in solving sociological tasks and modelling economic problems. It makes sense to speak about FEM only in connection by deploying it on computers - it can be boldly said that the method is the product of the period of modern computers. A current interesting thing is that it is for the scientific-technical calculations of meteorologists, and has the greatest requirements for computer size and speed so that a weather forecast for the entire earth could be calculated up to two hours after the gathering of measured data.

The finite element method (FEM) is presently considered the most effective numerical method for solving the problems of mathematical physics, that is the problems of prescribed differential or integral equations of the most various types, the systems of these equations, using variation non-equations, and other types. Typical examples are partial differential equations describing electric, magnetic or gravitational potential, Schrödinger’s equation, heat conductivity equations, the system of Maxwell’s equation, non-equations for the contact of two elastic-plastic bodies, etc.

The name of the method emphasizes the fact that a basic building stone is the element of finite dimensions differing from the infinitesimal view, which comes from balancing infinitely small elements.

Applying the FEM possibilities has lead to the storming development of software in solving a varied spectrum of technical and scientific tasks. All computer processes can also be basically automated, that is in using generating triangulation, setting up stiffness matrices, solving arising systems of algebraic equations, the graphic illustration of results, etc., and it is possible to entrust it to computers. A further big FEM preference is it perfectly enables descriptions in examined areas, which would not be possible with classic methods (for example, the method of accumulation or the screen method). For theoretical numerical analysis (existence solutions, convergence proofs, fault estimates, etc.) it is suitable that FEM
mostly relies on physical variation principles. It also enables us to use the effective tools of functional analysis and also it does not require sufficient assumptions for higher fineness solutions as for the classic method.

### Summary
The finite element method (FEM) is relatively old, a huge boom was brought about with PC expansion. The method is used for solving a huge spectrum of engineering tasks thanks to a whole serious of specialized programmes.

### 2.2. The finite element method in practice

#### Study time: 0.5 hours

#### Concepts to remember
- Variation principles, numerical methods, the preprocessing, processing and postprocessing of peripheral conditions, mesh density, element types

#### Lecture

- **The finite element method as a scientific branch**
  
  FEM is presently a widely and thoroughly elaborated scientic branch containing these parts:
  - **theoretical**: formulation of variation principles, deriving relations for various types of elements, etc.
  - **mathematical**: the problems of suitable numerical methods, the selection of algorithms, searching for fast algorithms (for example. 2,5D methods), existence proofs and convergence solutions, estimations of solution errors, etc.
  - **computer**:
    - preprocessing - generating input data, graphic illustration of classifying, input data, peripheral conditions, loading, data correction and arrangement etc.
    - processing - calculating element matrices, setting up whole system matrices, setting up matric equations and their solutions, etc.
    - postprocessing - calculating dependent parameters, output files, graphically illustrating results, periphery result outputs, etc.
  - **engineering problems**: the use of FEM possibilities for specific engineering tasks, that is separating bodies into elements – the selection of screen density, the selection of element type, the selection of a suitable element for a given task, entering necessary input data (peripheral and computer conditions), the selection of the forms of outputs (iso-surfaces, graphs), etc.
Let’s leave the first two parts to mathematicians, the third part is dealt with by the programmers of companies developing simulation software. Yet the last part in the previous listing will fulfill this study branch. The application of FEM programmes for real technological problems is often very demanding, it cannot be bypassed without deep knowledge of appropriate scientific branches (searching for and defining the links between individual inputs), but it is also necessary for understanding the basic principles of the method (to ability to estimate calculation mistakes and other areas)

- **FEM usefulness in practice**

FEM is very closely connected to information technology and software engineering. Its robustness and universality is conditioned on the unprecedented processing of data and the claims of the number of operations. The use of FEM in manual (without computer support) calculations is practically impossible. The program apparatus of the finite element method usually has two basic types of programmes:

- A program carrying out the calculation itself - numerical *core*
- Programs for preparing input data and processing the results - *pre-procesor* and *post-procesor*

The main requirements placed on numerical cores are: facility, reliability, robustness, output:

- **Facility** needs user requirements to implement tasks in the program, which a user needs. This requirement is fulfilled either by attempting maximal universality, or conversely by narrow specialization.

- **Reliability** means that all parts of the program are verified and tested and are physically and mathematically correctly implemented. One of the attributes of reliability is long-term development and feedback between users and program producers. Practical experience has shown that during good computer design it is possible to maintain numerical cores and many decades of development. The most implemented procedure is verified in such a product not only at the producers, but also by dozens and hundreds of calculations by program users.

- **Robustness** on one hand means the quality of an individual code, the minimalization of the occurrence of programmer errors, and one the other hand a clear and understandable interface design, which minimalizes the danger of misunderstanding between a programmer and a user, an understandable warning system and error reports, sufficiently thorough reports on tasks and not least top-quality documentation.

- **Output** is a primary requirement, but does not have to be reached at the price of compromising the fulfillment of the previous three. The demandedness of FEM calculations grows roughly with the squared to cubed power of the task range, so that programme performance together with the output of the used information technology are often a limiting factor, which determines the conception of FEM modelling.

The requirements for pre- and post-procesors are the most varied and more dependent on the branch and type of task. In engineering applications FEM in the present standard is the **support of geometric modelling** and the **automatized generation of FEM screens**
into a geometric template. Common requirements are the **imports** of geometric models from CAD programs. In several cases it leads to the close connecting of CAD programs with FEM preprocessors and numerical cores, so that the difference between CAD programs and FEM reduces. In the area of post-processing there is the obvious requirement of plotting resultant fields in the different variants of colourfully graduated iso-surfaces, generating the animation of deformation development and other quantities, illustrating the dependence of resultant qualities at a time at a given location or in a position along a defined curve into graphs. The standard is the transformation of the compounds of calculation vectors and tensor fields into selected co-ordinate systems. The processing of computer data, for example from the point of view of microstructure development, is usually solved outside the framework of a standard postprocessor, but a number of FEM program apparatuses offers well integrated modules.

- **Myths and superstitions about FEM**

  From a mathematical point of view the finite element method (FEM) represents

  - in a wider sense a numerical method of solving (partial) differential equations (in a continuum)
  - in a narrower sense only a discretizing technique of the definition field of the sought function (the continuum discretization method). The basis of the solution itself is then from several variation methods.

  In technical practice it has been shown that FEM is very strong in solving the tasks of the mechanics of flexible bodies. In the course of the last third of the twentieth century FEM was almost a monopolizing means in the numerical analysis of the mechanical systems of flexible bodies. It is implemented in a number of engineering program means: in purely analytical applications (traditional „big FEM packages” ADINA, NASTRAN, ANSYS, ABAQUS, MARC...), in programmes specialized on various specific technical problems - accident simulations, technological process simulation (a number of Pam programs, for example. Pam-Crash, Pam-Stamp, Pamp-Cast, specifically then in forming the programs Deform, Forge, FormFEM, etc.) and also finally in CAD systems as a means for fast design calculations (Pro-Engineer, Pro-mechanika ...). In the last two cases FEM itself often is covered “somewhere inside a program” and a user will not come into contact with it.

  In terms of industrial application a number a prejudices and myths have arisen in connection with FEM

  - **The most dangerous myth is a "blind" virus in that the calculated results are correct.**
  - FEM calculation is more trustworthy.
  - FEM calculation is quicker than analytic calculation.

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1At the present stage there exists FEM pre-processors equipped by geometric modellers using a supportive hierarchical model structure such as the system of individual parts (of separated bodies), which are described by a sequence of operations and they support inheritance, the possibility to suppress and renew individual properties and further conveniences usual in CAD. A series of CAD programs have modules for generating FEM screens, and for entering
• Introducing FEM in companies reduces development and design
• Introducing FEM leads to technological improvement.

These claims are not downright falsehoods, but are not universally valid:

• The person calculating guarantees the result correctness, never the method. It is possible to use this method as an unsuitable method such as from the point of view of the individual modelling of mechanical systems (for example, if decision-making uses a 1D, 2D or 3D continuum, moulds, girders, a solid-plastic or elastic-plastic material model), and also from the point of view of using calculated procedures - linear and non-linear, static or dynamic.

• For example, in the application of calculated results in estimating the life of cyclically loaded bodies there can be an analytical calculation, which corresponds to a given norm of calculating damage and the method, through which material data was gained more precisely and faster.

• Many times the calculations are faster, but model formation is very demanding.

• FEM in companies often touches upon purchasing programs. Using the program requires qualified staff. Of course, the management does not calculate with it. Sometimes it even happens that the program is not installed at all. It is more usually used at free moments, when staff successfully "tears it away" from more important duties. In such cases, it is cheaper not to buy the program, but the calculations.

• What is valid in the first point. A technologist guarantees quality calculations. On the basis of non-quality calculations it is not possible to improve technology.

It is generally valid that in industrial applications FEM is a means, and not an objective. From this point of view it is necessary to always consider not only whether to use FEM, but also how to use it.

Σ Summary

The problems connected with FEM are very extensive, you as a future user will be interested only in the engineering problem part. Fortunately we have a whole series of programs available on the basis on FEM and then only you prepare a screen in a pre-processor and enter initial and peripheral conditions in a post-processor, then you select and illustrate the results, which from the problem solution point of view has the best corresponding ability. The basic rule of your work then is to not blindly believe in the correctness of results.

2.3. The principle of the finite element method

Study time: 1 hour
The finite element method, on which most simulation programs today are founded, belongs to variation methods. These methods were formed in the mid-20th century with the appearance of Dirichlet’s principle for solving differential equations. The principle consists in the fact that for individual types of differential equations it is possible to set up Dirichlet’s integral, whose minimalization is solved by a given equation. By finding a function, which minimalizes this functional, a solution is found for a given differential equation with known peripheral conditions.

With variation methods we look for a solution to a given task using an experimental design. We proceed so that we express a given functional as a function of a proposed experimental design. From all possible solutions, fulfilling peripheral conditions, we then select that which is a given stable functional - it ensures its minimum. We call a variation principle a mathematical procedure, which enables the selection of a problem solution from a whole class of possible solutions.

- The idea of discretization

In most cases we need to find such variables, such as for example there are displacement, tension, temperature, pressure and speed, which are functions of co-ordinates x, y, z. In the case of a transfer or unstable problem these variables are functions not only of co-ordinates x, y, z, but also time t. The geometry of the problem is very often irregular.

A basic step of the finite element method is to divide (discretize) any mechanical system (a beam, a frame, a grate, a board, a wall, a block...) into a finite number of elements which are usually simple and geometric (line segments, triangles, rectangles, prisms, ...). The division of elements in any case is not unequivocal and is strongly affected by technical experience and the feeling of the researcher viz Fig. 2.1.

### Concepts to remember

- Element, centre point, discretization, screen, functional, variation, base function, stiffness matrix
Fig. 2.1 An example of screening a slab during the simulation of vertical rolling, in a location where we expect the greatest form change the finest screen is used (screen size 0,5 mm).

There exists today the automatic screen generators of finite elements, but for a problem a researcher always has the „last word“ (correcting and adjusting a formed screen in corners, transfers, etc.)

If we then divide this system into elements, we especially select for determining (the first unknown) only those points of a structure (on the basis of an infinite number of points in a continuum), which most often lies in the corners of elements, finite points, the centres of edges, surfaces, etc.). In these centre points (for which there is a finite number) we calculate primary unknown quantities (in static tasks usually of displacement) and from them derive secondary ones (in the statics of internal forces and tension). From the known functional quantities of the centre point values of one element we then can determine any quantity of any element point.

- **Basic element form**

The form, size and number of elements is necessary to carefully select so that an original area is divided the best, that is for a screen area to most closely imitate an original model. The shape of an element is often derived from the geometry of a screen area. If the geometry, material properties and resultant variables can be described by a single co-ordinate, then it is possible to use bar-shaped elements (Fig. 2.2.a). It is possible to use these elements for example in modelling temperature distribution in a bar or modelling bar deformation during axial loading. Even if these elements have a certain cross-section, they are often schematically indicated only as line segments (Fig. 2.2.b).

![Fig. 2.2. a) a bar-shaped element  b) schematic indication of a bar-shaped element](image)

In the case that a configuration and the other properties of a solved problem can be described by two independent co-ordinates, then it is possible to use 2D elements (Fig. 2.3.). Basic used elements in 2D analysis are usually a triangular elements.
Fig. 2.3. Various types of elements for the creation of a 2D screen

triangular element  rectangular element  equalateral element  parallelogram element

A four-sided element (and the derived squared element from it or a parallelogram) can be composed of two or four triangular elements, as shown in Fig. 2.4. In some cases the use of four-sided elements is more suitable.

Fig. 2.4. Four-sided elements composed of two or four triangles

Node 1, 2, 3, 4

In the case that the properties are the functions of three co-ordinates, then it is necessary to use 3D elements (Fig. 2.5.). The basic element of a 3D screen is usually tetrahedral. In some cases it is possible to use tetrahedrals consisting of five three-sided pyramids.
Fig. 2.6. *Various types of three-dimensional elements*

*four-sided right-angled prism six-sided*

- **Element size**

  The element size directly affects solution convergence and that is why it has to be selected with deliberation. If the size of an element is small, a more precise solution is then expected. The smaller size means an increase in calculation time. That is why there is often used elements of various sizes at the same volume. In locations, where a greater gradient of variables is expected then used for a more fine screening.

- **The number of elements**

  The necessary number of elements for discretization is connected with a required exactness, the size of elements and the included number of degrees of freedom. More elements usually indicate a more precise solution, but for every problem there exists a certain number of nodes $N_0$, over which the exactness of a solution doesn’t increase. A graphic illustration is shown in Fig. 2.8.

![Graph showing the effect of increasing the number of elements on solution exactness](image)

**Fig. 2.8. The effect of increasing the number of elements on solution exactness**

*exact solution solution gained using FEM the number of elements*

- **Element form**

  With a too-deformed shape we get a badly conditioned element matrix (viz. below), which can lead to local errors. From this point of view an ideal shape is in the area of a cube, in equations then in a square, or an equalateral triangle.

  In generating a screen the approach of elements to an ideal shape is evaluated by means of the size of internal angles, which press sides together resp. the element walls or by using the ratio between the surface and the periphery (by the volume and content of a casing) of an element. An example of a good and bad triangular 2D element is shown in Fig. 2.9. In Fig.
there is then illustrated the *surface shape factor*, which for the qualitative analysis of a screen uses the program FORGE 3D (it is not recommended to use a screen, when this factor falls below 0,4).

![Surface Shape Factor](image)

**Fig. 2.9. Good and bad form of finite element i**

**Obr. 2.10. Evaluating the quality of a screen in the programu FORGE**

- **Courant base function**

Using the finite element method we look for an approximate solution using Poisson’s equation. The sought function has same common shape $u(x)$ (the blue curve in **Fig. 2.11.**). An approximate solution has then the shape along time linear function $u_i(x)$ (the red curve in **Fig. 2.11.**), which is connected in the elements (between the individual centre point meshes). From **Fig. 2.11.** it is evident that accuracy depends on the number of nodal points (resp. on element size).
Fig. 2.11. Approximate solution gained using the finite element method

The exceptionality of the finite element method consists in a genial structure along a time linear function, which is possible to describe using the following equation:

\[ u_i = \sum_{i=1}^{N} d_i \cdot \varphi_i \]  

(2.1)

The function \( u_i \) we then record using Courant’s base function \( \varphi_i(x) \), which gets its name in honour of the discoverer of FEM. \( N \) is the number of elements and \( d_i \) is an unknown constant.

The shape of the base function is illustrated in Fig. 2.12. Thus base function \( \varphi_i(x) \) has node \( i \) and the value of 1 in other centre points then the value 0.

Fig. 2.12. Base function shape
**Example 2.1**

**Determining the equation of a base function**

We have a round bar with length $l$ and cross section $a$. The bar is divided into $N$ elements of the same length.

The length of each element is thus in this case $\frac{1}{N}$, which means $x_1 = 0$, $x_i = \frac{1}{N}$, $x_2 = \frac{2}{N}$, $x_i = \frac{i}{N}$.

Function $\varphi_1$ is linear in an interval from $x_i$ up to $x_2$ and it is thus possible to record it this way:

\[
\varphi_1 = a \cdot x + b \\
\varphi_1(x_1) = 1 \Rightarrow a \cdot x_1 + b = 1, \Rightarrow b = 1 - a \cdot x_1, \\
\varphi_1(x_2) = 0 \Rightarrow a \cdot x_2 + b = 0,
\]

From the first equation we can state $b$ and install it a second time. Thus we gain the relation:

\[
a \cdot x_2 + 1 - a \cdot x_1 = 0 \Rightarrow a = \frac{1}{x_1 - x_2} \text{ and from it implement value } b:
\]

\[
b = 1 - \frac{1}{x_1 - x_2} \cdot x_1,
\]

\[
\varphi_1(x) = \frac{x}{(x_1 - x_2)} + \left(1 - \frac{x_1}{x_1 - x_2}\right) = \frac{x - x_1}{x_1 - x_2}.
\]

We crop the equation of this function so that it is valid only in intervals $x_i, x_2$:

\[
\varphi_1(x) = \max \left(0, \frac{x}{x_1 - x_2}\right)
\]

By attaining $x_2 - x_1 = \frac{1}{N}$ we get a general equation for all base functions and for all $x$:

\[
\varphi_i(x) = \max \left\{ 0; 1 - \frac{x - \frac{i+1}{N}}{\frac{i}{N} + \frac{i+1}{N}} \right\}
\]
• **Functional**

Let’s now return back to unknown function $u$. We look for this function using a variation of a functional, which represents the energy system needed for attaining displacement $u$.

**What is a functional?**
I suppose that the concept functionary is unknown to you. And thus what the concept function is.

**So what is a function?** It is a concept, which you use so often that it can be difficult for you to define it briefly and clearly.

**So thus a function is such a rule (usually matematical), which sorts the element (of a number) of one power to an element (of a number) from a second power.**

A functional is then a rule, with which a function sorts a number. A typical representative of a functional is a certain integral, which numerically represents the area under a curve.

**Variation is for a functional what derivation is for a function.**

The following functional represents energy system $J$ necessary for certain displacement $u$, in the case of the problem of 1D elasticity (bar with a length 1 m) described in Chapter 1:

$$
\int_0^1 \frac{1}{2} \cdot E \left( \frac{du}{dx} \right)^2 \cdot dx - \int_0^1 f(x) \cdot u(x) \cdot dx - P \cdot u(1) = J
$$

(2.2)

The first element of equation (2.2) represents the work of internal tension forming in the bar, the element then represents the work of the forces in nodes and the third member represents work at the end point of the bar, on which there acts external tension $P$.

From the law of metal flow using the way of least resistance it is valid that the energy for displacement has to be minimal. It is thus valid that if you find the minimum of a functional so you will gain the function $u$, solved in equation (1.7). We gain the minimum of a functional so that we place its variation equal to zero:

$$
\int_0^1 E \cdot \frac{du}{dx} \cdot \delta \left( \frac{du}{dx} \right) \cdot dx - P \delta u|_{x=1} - \int_0^1 F \cdot \delta \cdot u \cdot dx = 0
$$

(2.3)

With the use of equation (2.1) it is possible to transfer this equation into the following form:

$$
\int_0^1 E \cdot d \left( \sum_{i=1}^N d_i \cdot \varphi_i \right) \frac{d}{dx} \cdot \varphi_i \cdot dx - \int_0^1 f(x) \cdot \varphi_i = 0
$$

(2.4)

In fact equation (2.4) is represented by the following system of equations:
It is possible to symbolically write this system of equations using a matrix number so that:

\[
\begin{bmatrix}
    d_1 \\
    d_2 \\
    \vdots \\
    d_N
\end{bmatrix}
\begin{bmatrix}
    \varphi_1 \\
    \varphi_2 \\
    \vdots \\
    \varphi_N
\end{bmatrix}
= 
\begin{bmatrix}
    f(x) \varphi_1 \\
    f(x) \varphi_2 \\
    \vdots \\
    f(x) \varphi_N + P
\end{bmatrix}
\]

or

\[ K \cdot D = F \]  \hspace{2cm} (2.6)

Where 
- \( K \)…global matrix of stiffness  
- \( D \)…global vector of unknown parameters  
- \( F \)…global vector of loading

As we see in the equations (2.3 to 2.5) there is found unknown function \( u \) in the form of a derivation according to \( x \). Because we will need to know the **derivation of the base function** for further calculations. We determine derivation as a tangent of angles, in which function \( \varphi_i(x) \) is pressed together with axis \( x \) (viz. **Fig. 2.13**).

**Fig. 2.13. Determining the derivation of a base function**

at interval \((x_{i-1}, x_i)\) (when \( x_i - x_{i-1} = 1/N \)) the tangent of the angle is determined as:

\[
tg \alpha = \frac{1}{1/N} = N
\]  \hspace{2cm} (2.7)

At interval \((x_{i+1}, x_i)\) then in such a way:

\[
tg \alpha = \frac{1}{-1/N} = -N
\]  \hspace{2cm} (2.8)

You can generally write:
\[
\frac{d\varphi_i}{dx} = \begin{cases} 
N & x \in (x_{i-1}, x_i) \\
-N & x \in (x_i, x_{i+1}) \\
0 & x \not\in (x_{i-1}, x_{i+1})
\end{cases}
\] (2.9)

- Matrix of stiffness

The matrix of system \( K \cdot D = F \) has an advantageous track structure, which together with other numerical properties, such as positive definiteness (symmetrical with its own positive numbers), contributes to effective solvability to very extensive problems as well.

The unknown parameters of vector \( D \) can be laid out so that the coefficients are a distributed matrix \( K \) in a relative narrow strip around the main diagonal (viz. Example 2.2.)
Example 2.2

1D problem of elasticity – solving using FEM

We have a round bar with length \( l = 1 \) m and cross section \( a \). The bar is divided into 5 elements of the same length.

By using equation (2.5) we calculated all elements of the matrix of stiffness. The stiffness matrix has 5 rows and 5 columns, thus in total 25 elements.

\[ \begin{bmatrix} (1) & (2) & (3) & (4) & (5) \\ (6) & (7) & (8) & (9) & (10) \\ (11) & (12) & (13) & (14) & (15) \\ (16) & (17) & (18) & (19) & (20) \\ (21) & (22) & (23) & (24) & (25) \end{bmatrix} \]

Calculation of element (1):
\[
\int_0^1 \frac{d}{dx} \varphi_1 \cdot \frac{d}{dx} \varphi_1 \cdot dx = \left[ (-N) \cdot (-N) \cdot dx \right]_0^1 + \left[ 0 \cdot 0 \cdot dx \right]_0^1 = \int_0^1 N^2 \cdot dx = \int_0^1 \frac{x^2}{x} \cdot dx = \frac{x^2}{x} \bigg|_0^1 = N
\]

Calculation of element (2):
\[
\int_0^1 \frac{d}{dx} \varphi_2 \cdot \frac{d}{dx} \varphi_1 \cdot dx = \left[ (N) \cdot (-N) \cdot dx \right]_0^1 + \left[ (-N) \cdot 0 \cdot dx \right]_0^1 + \left[ 0 \cdot 0 \cdot dx \right]_0^1 = \int_0^1 (-N^2) \cdot dx = \int_0^1 \frac{x^2}{x} \cdot dx = \frac{x^2}{x} \bigg|_0^1 = -N
\]

Calculation of element (3):
\[
\int_0^1 \frac{d}{dx} \varphi_3 \cdot \frac{d}{dx} \varphi_1 \cdot dx = \left[ 0 \cdot (-N) \cdot dx \right]_0^1 + \left[ (N) \cdot 0 \cdot dx \right]_0^1 + \left[ (-N) \cdot 0 \cdot dx \right]_0^1 + \left[ 0 \cdot 0 \cdot dx \right]_0^1 = 0
\]

Due to the special structure of base functions elements (4) and (5) will also be equal to zero.
**Example 2.2 - continuation**

The calculation of element (6):

\[
\int_0^1 \frac{d}{dx} \varphi_1 \cdot \frac{d}{dx} \varphi_2 \cdot dx = \int_0^1 (\frac{d}{dx} N) \cdot (\frac{d}{dx} N) \cdot dx + \int_0^1 0 \cdot (\frac{d}{dx} N) \cdot dx + \int_0^1 0 \cdot 0 \cdot dx =
\]

\[
= \int_0^1 N^2 \cdot dx = \left[ N^2 \cdot x \right]_0^1 = N^2 \cdot (x_2 - x_1) = -N^2 \cdot \frac{1}{N} = -N
\]

The calculation of element (7):

\[
\int_0^1 \frac{d}{dx} \varphi_2 \cdot \frac{d}{dx} \varphi_2 \cdot dx = \int_0^1 N^2 \cdot dx = \left[ N^2 \cdot x \right]_0^1 = N^2 \cdot \frac{1}{N} = 2N
\]

The calculation of element (8):

\[
\int_0^1 \frac{d}{dx} \varphi_3 \cdot \frac{d}{dx} \varphi_2 \cdot dx = \int_0^1 (\frac{d}{dx} N) \cdot dx + \int_0^1 (\frac{d}{dx} N) \cdot dx + \int_0^1 (\frac{d}{dx} N) \cdot dx + \int_0^1 0 \cdot 0 \cdot dx =
\]

\[
= \int_0^1 N^2 \cdot dx = \left[ N^2 \cdot x \right]_0^1 = N^2 \cdot (x_2 - x_1) = -N^2 \cdot \frac{1}{N} = -N
\]

It is calculated similar for other elements. You can try yourself to complete the sum of any element of our stiffness matrix. The resultant matrix thus looks in such a way:

\[
\begin{bmatrix}
N & -N & 0 & 0 & 0 \\
-N & 2N & -N & 0 & 0 \\
0 & -N & 2N & -N & 0 \\
0 & 0 & -N & 2N & -N \\
0 & 0 & 0 & -N & N
\end{bmatrix}
\]

The computer then solves this matrix using several numerical methods for solving systems of linear equations, for example, using Gauss’s elimination method.

- **The effect of numbering centre points on the shape of a stiffness matrix**

  The structure of matrix \( K \) is connected to numbering elements and screen nodes. For 1D problems the numbering of nodes is very simple, but for modelling in 2D the situation is complicated. Let’s compare two various numberings for planar problems, which are presented in the following two models (Fig. 2.14.).
Each element has three 3 nodes = 6 unknown (in each node there are 2 unknown shifts (in the direction of axes x and y)) thus the elemental stiffness matrix will be of an order 6 x 6. For the whole body there are unknown 12 together (in each node 2 shifts) and the global stiffness matrix will be of an order 12 x 12.

**Numbering a)**

The stiffness matrix of the first element due to the expansion to a dimension 12x12 (that is for the dimension of global matrix K) contains a non-zero contribution at positions at 1., 4., 7. and 8. of rows and columns:

\[
\begin{bmatrix}
  k_1 & k_1 & k_1 & 0 & 0 & k_1 & k_1 & 0 & 0 & 0 & 0 & 0 \\
  k_1 & k_1 & k_1 & 0 & 0 & k_1 & k_1 & 0 & 0 & 0 & 0 & 0 \\
  k_1 & k_1 & k_1 & 0 & 0 & k_1 & k_1 & 0 & 0 & 0 & 0 & 0 \\
  k_1 & 0 & k_1 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Thus by totalling up the expanded matrix of all four elements we attain global matrix K:
Strip width = 8

Indices indicate the number of elements that contribute their stiffness to the resultant stiffness in matrix $K$. 
Numbering \textit{b})

Global matrix $K$ will now have such a structure:

Strip width = 6

$$
K = \begin{bmatrix}
    k_1 & k_1 & k_1 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    k_1 & k_1 & k_1 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    k_{12} & k_{12} & k_{12} & k_{12} & k_2 & 0 & 0 & 0 & 0 & 0 \\
    k_{12} & k_{12} & k_{12} & k_{12} & k_2 & 0 & 0 & 0 & 0 & 0 \\
    k_{23} & k_{23} & k_{23} & k_{23} & k_{23} & k_3 & 0 & 0 & 0 & 0 \\
    k_{23} & k_{23} & k_{23} & k_{23} & k_{23} & k_3 & 0 & 0 & 0 & 0 \\
    k_{34} & k_{34} & k_{34} & k_{34} & k_{34} & k_{34} & k_4 & 0 & 0 & 0 \\
    k_{34} & k_{34} & k_{34} & k_{34} & k_{34} & k_{34} & k_{34} & k_4 & 0 & 0 \\
    k_{4} & k_{4} & k_{4} & k_{4} & k_{4} & k_{4} & k_{4} & k_{4} & k_4 & 0
\end{bmatrix}
$$

Nodes can optimally be numbered so that the width of the matrix strip is minimal. It is necessary from the point of view of the amount of stored data on the effectiveness of the system solution. The number of operations and thus the length of calculations are linearly dependent for the direct method (GEM) on the number of unknowns and quadratically on the width of the strip.

The minimalization of the strip for all commercial systems of FEM is automatically ensured by using a program procedure.

- \textbf{Solving a linear equation system}

\textbf{Example 2.3}

\textbf{Solving a linear equation system}

Let’s solve the system graphically:

$$
\begin{align*}
3 \cdot x_1 + 2 \cdot x_2 &= 18 \\
- x_1 + 2 \cdot x_2 &= 2
\end{align*}
$$

The solution is the intersection of straight lines:

The solution is the intersection of straight lines: $x_1 = 4; x_2 = 3$. 

Siřka pásu = 6
In addition to the above-mentioned case these cases can generally set in (Fig. 2.15.).

Fig. 2.15. Examples of solving the intersection of two straight lines

*It has no solution*  *Infinitely many solutions*  *A wrongly conditioned system*

**Direct methods – Gauss’s elimination method (GEM)**

It deals with a method leading to (at least theoretically) a precise solution eventually using many steps. By row (never column) corrections we transfer this matrix into a form, when only zero is found under the main diagonal. The corrected matrix then corresponds to the system of equations, which is equivalent to an original system. The solution consists of two steps:

1. elimination (the transfer to a step form, creating a diagonal matrix)
2. reverse attainment (by numbering out the unknowns)

$$\begin{bmatrix}
    a_{11} & a_{12} & a_{13} & b_1 \\
    a_{21} & a_{22} & a_{23} & b_2 \\
    a_{31} & a_{32} & a_{33} & b_3
\end{bmatrix}
\downarrow
\begin{bmatrix}
    a'_{11} & a'_{13} & b'_1 \\
    a'_{22} & a'_{23} & b'_2 \\
    a''_{33} & b''_3
\end{bmatrix}
$$

Eliminace

$$x_3 = b''_3/a''_{33}$$

$$x_2 = (b'_2 - a'_{23} \cdot x_3)/a'_{22}$$

$$x_1 = (b_1 - a_{12} \cdot x_2 - a_{13} \cdot x_3)/a_{11}$$

Zpětné dosazení

Elimination  Reverse attainment
The number of operations is known for the direct method beforehand, for GEM the number is a proportional $2n^3/3$.

**Table 2.1** The number of operations for GEM depending on the number of unknowns

<table>
<thead>
<tr>
<th>$n$ – no. of unknowns</th>
<th>Elimination</th>
<th>Reverse attainment</th>
<th>In total</th>
<th>$2n^3/3$</th>
<th>From this elimination in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>705</td>
<td>100</td>
<td>805</td>
<td>667</td>
<td>87,58</td>
</tr>
<tr>
<td>100</td>
<td>671 550</td>
<td>10 000</td>
<td>681 550</td>
<td>666 667</td>
<td>98,53</td>
</tr>
<tr>
<td>1 000</td>
<td>$6.67 \times 10^8$</td>
<td>$1 \times 10^8$</td>
<td>$6.68 \times 10^8$</td>
<td>$6.67 \times 10^8$</td>
<td>99,85</td>
</tr>
</tbody>
</table>

A badly conditioned system for solving using GEM

Badly conditioned matrices are formed for faulty finite elements, which do not fulfill the criteria for an element’s shape (viz. Chapter Element shape).

Let’s solve a badly conditioned equation system using a calculator or PC:

$$0,0003 \cdot x_1 + 3 \cdot x_2 = 2,0001$$

$$x_1 + x_2 = 1$$

The exact solution: $x_1 = 1/3$ a $x_2 = 2/3$.

Calculation:
From the equation state we enter $x_2$ into the another:

$$x_1 = \frac{2,0001 - 3 \cdot (2/3)}{0,0003},$$

Of course the result depends on how many decimal places are stated for $x_2$

<table>
<thead>
<tr>
<th>Number of decimal places</th>
<th>$x_2$</th>
<th>$x_1$</th>
<th>Error in % for $x_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0,6670000</td>
<td>-3,330000</td>
<td>1099</td>
</tr>
<tr>
<td>4</td>
<td>0,6667000</td>
<td>0,000000</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>0,6666700</td>
<td>0,300000</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>0,6666670</td>
<td>0,330000</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0,6666667</td>
<td>0,333000</td>
<td>0,1</td>
</tr>
</tbody>
</table>

A badly conditioned matrix is formed for faulty finite elements which do not fulfill criteria for an element’s shape (viz. the chapter Element form). The effect of rounding off mistakes is necessary to be taken care of on a PC. It is carried out using the selection of the main elements (pivoting).

It is also possible to correct bad conditionality by using a multiple of all equations with a suitable number (scaling). Thus the determinant of the matrix of a system, which almost gets larger closer to zero.
It is further possible to carry out preconditioning, that is transferring a system of equations to a better conditioned system.

**Gauss-Seidl method**

In addition to direct methods iteration methods are used, for example. the Jacob or the Gauss-Seidl method. Let’s solve an iteration system using the G-S method.

\[
\begin{align*}
3 \cdot x_1 + 0,1 \cdot x_2 - 0,2 \cdot x_3 &= 7,85 \\
0,1 \cdot x_1 + 7 \cdot x_2 - 0,3 \cdot x &= -19,3 \\
0,3 \cdot x_1 - 0,2 \cdot x_2 + 0,1 \cdot x &= 71,4
\end{align*}
\]

From the first equation we can state the first unknown:

\[
x_1 = \frac{7,85 - 0,1 \cdot x_2 + 0,2 \cdot x_3}{3}
\]

From the second equation we can state the second unknown:

\[
x_2 = \frac{-19,3 - 0,1 \cdot x_1 + 0,3 \cdot x_3}{7}
\]

From the third equation we can state the third unknown:

\[
x_3 = \frac{71,4 - 0,3 \cdot x_1 + 0,2 \cdot x_2}{10}
\]

Thus we have attained the iteration rule for unknowns \(x_1, x_2, x_3\).

**1. iteration**

As an initial estimate of results we place all unknowns equal to zero and we will further make this estimate more precise. We can use the iteration rule:

\[
x_1 = \frac{7,85 - 0 + 0}{3} = 2,616667
\]

If we state \(x_2\), we can already install for \(x_1\) the new approximation 2,6167:

\[
x_2 = \frac{-19,3 - 0,1 \cdot (2,616667) + 0}{7} = -2,794524
\]

If we state \(x_3\), we can already install for \(x_2\) the new approximation -2,794524:

\[
x_3 = \frac{71,4 - 0,3 \cdot (2,616667) + 0,2 \cdot (-2,794524)}{10} = 7,005610
\]

**2. iteration**

This procedure is further repeated until a certain number of iterations are unknown values numbered by their required accuracies. This phenomenon is called a convergence solution. If conversely distance is made for an exact solution it is called a divergence solution. (viz. **Fig. 2.16**).
Fig. 2.16. Examples of convergence (on the left) and divergence (on the right) solution
Summary

It is possible to summarize the FEM principle in this way: The whole Ω area discretizes on a screen of finite elements. The original Ω often has a complicated shape and the models parameters are not constant along the entire Ω. As opposed to it each element has a simple boundary and the model parameters can be taken as constants. As a final consequence instead of one partial differential equation, we solve using a system of linear equations with $n$ unknowns, which is possible to solve on a computer using numerical methods.

It is necessary to keep in mind that such a solution is always loaded with errors. It depends mainly on the quality of the screen and on the simulation program itself (its robustness) as to what size the errors will be and it depends on the service of the program to possibly identify and minimalize them.

Questions for Chapter 2

2.1. What is the basis of the variation method?
2.2. What is meant under the concept discretization?
2.3. Who forms the finite element model mesh?
2.4. How to distinguish basic types of elements?
2.5. What size of elements affect solution results?
2.6. What are the ideal forms of elements?
2.7. How is the quality of a screen evaluated?
2.8. What form does the sought-after function have in solving using FEM?
2.9. What is a base function? What are its qualities?
2.10. What is a function?
2.11. What is a functional?
2.12. How do we determine the minimum of a functional?
2.13. At what interval does the derivation of the base function $\phi_i$ of a non-zero value gain?
2.14. What properties does a stiffness matrix gained using FEM have?
2.15. How do you look for the unknown constants in a stiffness matrix?
2.16. What is understood under the concept of a wrongly conditioned system of equations?
2.17. Can you describe the procedure for solving a system of equations using Gauss’ elimination methods?
2.18. What is the difference between an indirect and a direct method of solving a system of equations?
2.19. What is the convergence solution?
2.20. What can affect if a solution will converge or diverge?
Problems to be solved in Chapter 2

2.1. At interval $x_1$ up to $x_2$ determine base function equation $\phi_2$ for a situation as in Example 2.1.

2.2. Determine the derivation of base function $\phi_2$ from the previous problem at an interval $x_2$ up to $x_3$.

2.3. Draw a retangular object in a 2D area. Carry out its discretization using 6 triangular elements. Number the elements and centre points.

2.4. Numerouslly solve the system of equations from Example 2.3

2.5. Using Gauss’s elimination method solve this system of equations:

$$3 \cdot x_1 + 0,1 \cdot x_2 - 0,2 \cdot x_3 = 7,85$$
$$0,1 \cdot x_1 + 7 \cdot x_2 - 0,3 \cdot x = -19,3$$
$$0,3 \cdot x_1 - 0,2 \cdot x_2 - 010 \cdot x = 71,4$$

Other sources


1. METAL FORMING OPERATIONS AS A SYSTEM

Objective: After studying this chapter through you will be able to:

- Explain a systemic approach to metal forming.
- Identify all parts of the system of metal forming.
- Have an overview about variable systems and how to gain them

Content of Chapter 3

1.1. Introduction
1.2. Material variable
1.3. Tools and machines
1.4. Tribology
1.5. Metal flow

3.1. Introduction

Study time: 0.5 hours

Concepts to remember

System, tribology, process variables

Lecture

I.

Summary

The basic systemic approach is that all components of a system are evaluated together, we are interested in their interactions together and their effects on the results.

3.2. Material variables

Study time: 1 hour
For a given chemical material composition and deformation history, or the history of the heat treatment (of a microstructure) the most important material quantities are directly affecting metal flow: natural deformation resistance and formability, and both quantities can differ in various directions (anisotropy). They indirectly affect (across the lay-out of the thermal field of a semi-product) the thermal properties of metal flow: the coefficient of thermal conductivity $\lambda \, [\text{W.m}^{-1}.\text{K}^{-1}]$, specific thermal capacity $c \, [\text{J.kg}^{-1}.\text{K}^{-1}]$, density $\rho \, [\text{kg.m}^{-3}]$ and thermal longitudinal expansibility $\alpha \, [\text{K}^{-1}]$.

- **Natural deformation resistance**

  $\sigma_p = \sigma_i = \sqrt{\frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \right]}$, \hspace{1cm} (3.1)

  $\sigma_p = f(M,T,e,\dot{e})$, \hspace{1cm} (3.2)

  Where $M$ is the metallurgical character of a metal, $T$ is temperature, $e$ is deformation and $\dot{e}$ is deformation speed. This dependence is graphically most often presented as tension curve $\sigma$ - deformation $e$.

  The exact mathematical description of the curve $\sigma$-$e$ is one of the most important parts of the forming process. It is necessary when storing data into simulation programs. One decisive source are the outputs from plastometric experiments (tortion, pressure). During operation or the laboratory measuring of rolling forces there can also be gained only median natural deformation resistance.

  A mathematical description of a deformation tension curve is possible to carry out in a way which can be divided into three groups methods:

  - Every thermomechanic parameter is independently shown and its effect has been deeply analysed. They form complicated and long equations. We can describe a continuous tension value as a function of all variables by which each variable can contain in itself the influence of another variable. As an example for understanding let’s present the possibility of the deformation speed effect. With the above-mentioned temperature effect on deformation the basic relation can also have a speed exponent, in the form:
\[ \sigma_p = A_0 \cdot \dot{\varepsilon}_m \cdot \dot{\varepsilon}^{a-b T} \cdots, \]  

(3.3)

It is not difficult to also imagine the possibility of other effects on this exponent, for example structural and similar areas. Thus the equation grows to a length only for describing one parameter. Moreover, we can select a simple example of deformation speed, for the deformation effect is a much more complicated situation. We have to basically distinguish a deformation less than pico, greater than pico and a minimal deformation in a constant state. They are at least solvable with the addition of other elements such as a product into equation (2.5) and consequently it is a whole logarithmic equation, and this disproportionately complicates this.

- Experimental data from plastomers processed into polynoms. These equations are not common and in modern calculations, they are not used as a "nourishment" for FEM. In various forms there can be used the tabular results of plastometric tests, which enables the specific insertion of one specific experiment into a FEM calculation, but from the point of view of the entire database and user comfort this method is unreal. There has presently appeared the first signs of a solution using neural networks, which in principle enable a computer "to learn" to form from read values internal dependences, which then can be used for consequent calculations.

- Constitutive equations with the possibility of a complete physical approach or a partially phenomenological one. This approach consists in the registration of the real courses of curves gained from plastometers and by determining significant points on these curves and by their physical descriptions. It deals especially with pico points, during maximal tension and pika deformation there are determined the value \( \sigma_{\text{max}} \) which is pico maximal experimental tension, \( \varepsilon_p \) which is the pika maximal experimental deformation and of course continuous values of tension such as the deformation function in defined constant parameters (temperaturem deformation speed). It is possible to imagine, and such an equation exists that describes physically as well the point of a stabilized state such as tension size and deformation and one or more equations are used with limitations, but the validities of variable deformation are connected to each other.

- The greatest problem is to describe the effects of deformation. While the temperature or deformation-speed effect is monotoneous, it is not so or deformations. That is why the following analysis will be devoted mostly to an attempt to explain a method how to mathematically include the deformation effect into an equation. Real curve description \( \sigma-e \) can then have only the following shape for the effect of one parameter (here as deformation):

\[ \sigma_p = A_0 \cdot \varepsilon^n, \]  

(3.4)

\[ \sigma_p = A_1 \left(1 + A_2 \varepsilon\right)^n \]  

(3.5)

\[ \sigma_p = A_3 - (A_3 - A_4) \exp \left(-A_5 \varepsilon\right) \]  

(3.6)

or for the impossibility to also include the effect of dynamic plastification:
\[ \sigma_p = B_0 \left( B_1 + \varepsilon \right)^n - B_2 \varepsilon^{n_u}, \quad (3.7) \]

or also elsewise with the partial effect of deformation speed, but a parameter deformation on the inside:

\[ \sigma_p = C_0 + \varepsilon^{n_u} \left( C_1 + C_2 \ln \dot{\varepsilon} + \frac{C_3}{T} \right). \quad (3.8) \]

A still more complicated equation looks like this:

\[ \sigma_p = A \exp \left( m_1 T \right) \varepsilon^{n_m} \exp \left( \frac{m_2}{T} \right) \exp \left( 1 + \varepsilon \right)^{m_3 + T} \exp \left( m_7 \varepsilon \right) \varepsilon^{m_4} \varepsilon^{m_5}, \quad (3.9) \]

or under the time effect at a semi-interval:

\[ \sigma_p = \sigma_0 B_1 \exp \left( n_1 T \right) B_2 \exp \left( n_3 \varepsilon \right) B_3 \varepsilon^{n_6} B_4 T^{n_7}, \quad (3.10) \]

and also more complex:

\[ \sigma_p = A e^b \exp \left( -B \frac{\varepsilon}{\varepsilon_p} \right) \dot{\varepsilon}^{\frac{D_F}{T}} \exp \left( -G T \right), \quad (3.11) \]

where \( A, B, C, D, F, G, m, n \) are material constants and the entire equation is valid to the extent of deformation \( \varepsilon = 0 \) až \( \varepsilon_m \), where \( \varepsilon_m \) is deformation up to an inflex point for a pico. With the use of tension a corresponding stabilized state:

\[ \sigma_p = A \cdot \sigma_s \left[ \frac{\varepsilon}{\varepsilon_p} \exp \left( 1 - \frac{\varepsilon}{\varepsilon_p} \right) \right]^{d-f(T)}, \quad (3.12) \]

All equations of a similar type does not solve one basic question, how to fluidly describe a change in curve character at the inflex point with the possibility of least describing it into the stabilized metal flow, that is rather a parallel course of tension with a growing deformation.

**Metallurgical formability**

It deals with the ability of material to bear plastic deformation with disturbing cohesiveness under certain technological conditions (tenseness state, external friction). It depends on deformation conditions (temperature, deformation speed, the size of main tension, and the forming history) and material parameters (the chemical composition, the occurrence and character of precipitates, inclusions, cavities and initial microstructure). The formation and propagation of microcracks affects resultant metal flow. In algorithmic simulation programs the formation and propagation of a crack is controlled by fracture criterion. If it leads to some nodal finite element screen to exceed the critical value of fracture criterion, two cases can happen (according to the type of algorithm): screen fission, or element deletion. To
simplify it there is the more common kind of approach, where it is necessary to set up a sufficiently fine screen for big mistakes not to occur.

Fracture criterion

All criteria are set up on the assumption that damage forming during deformation leads to the formation of cracks. Most criteria is calculated as an integral, representing the time dependence of tension and deformation. A crack forms if the integral value exceeds critical value \( C \). In expert literature it is possible to find a whole series of formulae for criteria calculation, one of which is Oyan’s criterion:

\[
\varepsilon_i(t) \int_0^t \left( 1 + A \frac{\sigma_m}{\sigma_i} \right) d\varepsilon_i(t) \geq C ,
\]

where: \( \varepsilon_i \) – deformation intensity, \( \sigma_m \) – mean (hydrostatic) tension, \( \sigma_i \) – tension intensity

According to this formula in the material there is a loss of cohesion when exceeding critical value \( C \). In usual cases \( C \) is constant for a given material. It is usual to state \( C \) in this way:

\[
C = f (M, T, \dot{\varepsilon}) ,
\]

where: \( M \) - is the metallurgical character of a metal (chemical composition, purity, structure), \( T \) – is temperature, \( \dot{\varepsilon} \) – is the deformation speed.

Inversion analysis is used in literature for determining dependence (3.14). Thus it is the connecting of mathematical modelling using a programme on the basis of FEM and laboratory modelling on selected equipment. The principle is the following: using mathematical modelling an analysis of laboratory equipment is carried out. For each centre point there is determined criteria size, or other research parameters (temperature, deformation speed, etc.), the results are then compared with a real experiment which leads to crack formation. Using feedback, it is possible to transfer from a micro scale to a macro scale and gain information about crack formation in individual steps. On the basis of literary studies it is possible to say that the value of the critical value of fracture criteria grows in single-phase areas with increasing temperature and decreasing deformation speed.

Further in our work there will be used the Lathamo-Cockcroft criteria characterized by the following equation:

\[
\varepsilon_i(t) \int_0^t \sigma_1 d\varepsilon_i(t) \geq C ,
\]

where: \( \sigma_1 \) – the biggest main tension, \( C \) – critical value, \( \varepsilon_i \) - deformation.

Material thermal properties
Summary
We have become acquainted with a great number of equations for describing natural deformation resistance. Several of them are used in FEM commercial programs. It is possible to describe the level of material formability using fracture criteria. Basic thermal quantities have an effect on heat conduction: the co-efficient of thermal conductivity \( \lambda \), specific thermal capacity \( c \) and density \( \rho \). Reduced value \( \lambda \) and increased value \( c \) and \( \rho \) leads to great temperature differences between the surface and the centre of a semi-product. This difference in combination with thermal expandability causes thermal strain.

3.3. Tools and machines

Study time: 1 hour

Concepts to remember
Tool geometry, tool surface, tool motion

Lecture

The selection of equipment for a given process is carried out on the basis of knowledge of a machine’s energetic characteristics and also production accuracy. Variables characterizing tools are: design and geometry, surface quality, stiffness and mechanical and thermal properties under forming conditions, roll wear (a change of geometry against the design). In the case of rolling mill stands there enters into the game mainly roll jump/ displacement as a function of the rolling force.

Summary
Machine selection for a given process is carried out with knowledge of the energetic characteristics of machines and tool production accuracy. The variables characterizing tools are: design and geometry, surface quality, stiffness and mechanical and thermal properties under forming conditions, roll wear (a change of geometry against the design). In the case of rolling mill stands there enters into the game mainly roll jump/ displacement as a function of the rolling force.
3.4. **Tribology**

**Concepts to remember**

- Tribology, contact, friction, wear, lubrication, Coulomb’s Law, friction according to Trseco

**Lecture**

Tribology is a scientific branch, which deals with the behaviour of contacting surfaces in motion together or attempting motion together. Mutual links in a tribological system are shown in Fig. 3.6. The speciality of a tribological process during rolling in comparison with the contact of solid bodies in mechanics: they elastically deform rolls, plastically formed bodies; on the surface of a rolled product there is a roll impression; high pressure on contact surfaces (up to 500 MPa in hot rolling, up to 2 500 MPa in cold rolling); the size of contact surfaces is nor constant, it gradually increases.

**Tribologic process**

- Contact processes
- Process friction
- Process Wear
- Process Lubrication

**Surrounding processes**

- Technological process

**Other processes**

- Processes okoli
- Technologické procesy
- Další procesy

**Fig. 3.6. Mutual links in a tribologic system**

- **Contact processes**

  Mutual contact is a basic sign of a tribological system. It is necessary to consider the basic form-dimensional (viz. Fig. 3.7.) and material properties touching on the parts, which
have mutual connections and the reactions between them. Interaction can be material, physical, chemical, etc.

It is necessary to consider these effects:

- the number of bodies in contact,
- the macrogeometry and microgeometry of bodies,
- physical, chemical and mechanical properties,
- the kinds of deformation between bodies,
- the type and speed of relative motion.

Submicroscopic roughness

Microscopic roughness

Form and position deviation
Surface waviness
An ideally flat surface
• **Friction processes**

Mechanical friction is the result of shear contact between two surfaces, which are usually dry or lubricated. Friction prevents organized body motion and causes the dissipation of mechanical work with a non-reversible heat transfer. Friction between dry surfaces has an uncontinuous character.

**Wear processes**

Wear is an undesired change (loss) of surface or in the dimensions of solid bodies, caused either by mutually acting functional surfaces, or by functional surfaces and media, which causes wear. It is shown in the removal or shifting of mass particles from the functional surfaces by mechanical effects, and sometimes by the other effects that accompany it, for example chemical, electrochemical and other effects. There exists six basic kinds of wear: abrasive, adhesive, erosive, fatigue, cavation and vibration.

It is possible to describe measuring quantitatives using wear $W_s$ [mm$^3$/N.m] (defined as a volumic loss of material depending on the length of the contact and compressed force).

With wear in the mathematical analysis of rolling we can either meet it in the form of models applied to programs on a FEM basis, which are able to calculate the rate of machine wear, or in the form of changes in the initial geometry of a roll.

• **Lubrication processes**

If it comes from a basic tribological system, four basic states of friction can happen:
- The friction of dry bodies (dry friction)
- Liquid friction
- Fluent friction
- Plasmatic friction

Individual states occur in practice with limits, there often happens a combination of individual types. In slide friction two solid metals form according to Stribeck with a certain simplification into 4 types of friction:
- Dry friction is characterized by a high value of the friction coefficient
- Boundary friction forms between two sliding surfaces, between which there is a great thin molecular layer of lubricating film. It is characterized that in the increasing relative speed of friction surfaces there significantly reduces the friction coefficient. Boundary friction forms during heavy loading and a relatively low sliding speed.
- Hydrodynamic friction forms during a high relative slipping speed in which it is not possible to exactly determine the dependence between the friction coefficient and relative sliding speed. Friction forms by the acting internal friction of lubrication layers. For hydrodynamic friction the friction coefficient increases on very slowly depending on the relative sliding speed. Besides this the size of the friction coefficient is considerably dependent on the mean pressure of the tension of the lubrication film. The higher the surface pressure, the lower the friction coefficient is.
- Mixed friction is defined as a transit area between the boundary friction and hydrodynamic friction. The pressure of the transfering lubricating film of hydrodynamic friction is disturbed as a consequence of the tops of surface roughness of the friction
surfaces. Mixed friction mostly forms at low relative sliding speeds and high loading in thin fluid lubricating films or during insufficient lubricants on the surface areas.

There exists various methods of determining a friction model (during rolling): the advance method, the sample breaking method, the torsion motion method.

**Summary**

The problem of friction is mostly the source of errors in the mathematical modelling of forming processes. A whole number of factors enters into the game, which in mostly drastic cases are simplified into a mere coefficient knowledge (Coulomb) or friction co-efficient (Tresca) friction.

**3.5. Metal flow**

**Study time: 1 hour**

**Concepts to remember**

Law of metal flow, the way of least resistance, deformation resistance, friction

**Lecture**

According to the law of least resistance metal particles often shift in the direction of the lowest deformation resistance, but it is energetically the most advantageous. The course of metal flow affects the property and qualitative parameters of a product and force and energy requirements for the process.

Metal flow depends mostly on:
- the geometry and the method of tool motion,
- friction conditions,
- temperature and speed conditions in deformation zones (deformation resistance) and
- in the form of the solid ends of a semi-product.

The first three points have already been thoroughly analyzed in the preceding chapters. The solid end, that is the part, which does not very importantly lie in the area of the deformation zone of a semi-product, affecting the resultant metal flow during rolling. From technological work it is known that the beginning and end of rolling stock widens in a completely different way than rolling areas that are in a stable stage of rolling. In mathematical modelling there are thus necessary sufficiently long solid ends with an adequate density of the finite element screen. Moreover, solid ends affect the course of deformation in rolling stock even the mass itself, when during rolling there can be non-symmetrical rolling products leading to a twisting of a rolling product around their longitudinal axes.
Summary

Factors affecting metal flow have a direct effect on deformation resistance and also on the geometry of a semi-product and the tools and solid ends. The length of solid ends itself is often underestimated during simulation.

Questions for Chapter 3

3.1. Can you define the individual components of a metal formation system?
3.2. What affects all metal flow during forming?
3.3. How do you know material variables during forming?
3.4. How do you know process variables during forming?
3.5. What is anisotropy?
3.6. On what does natural deformation resistance depend?
3.7. What types of equations exist for describing natural deformation resistance?
3.8. Can you define metallurgical formability? How does it differ from technological formability?
3.9. What is a fracture criteria? How is it possible to determine its critical value?
3.10. How do you know the thermal properties of a material? Define them.
3.11. What thermal properties of a material influence heat conductivity?
3.12. What are the components of the tribologic process?
3.13. Can you characterize the semi-product surface from a shape and dimension point of view?
3.14. What parameters do we work with, if evaluating contact from a macroscopic point of view?
3.15. How is the friction coefficient designed?
3.16. What conditions lead to slide friction?
3.17. What model describes shear friction?
3.18. What friction model do you use in volumetric heat forming?

Problems to be solved in Chapter 3

3.1. Draw a scheme of a system for metal forming for drop-die forging.
3.2. Draw a curve describing deformation resistance in hot forming, define all connecting parameters.
Other sources


4. THERMOMECHANICAL ANALYSIS IN FORMING

**Objective:** After studying this chapter through you will be able to:

- Distinguish individual levels of thermomechanical analysis in forming
- Define the links between the individual variables of a system.
- Apply 4 steps for the successful use of simulation during forming analysis.

**Content of Chapter 4**

4.1. Introduction

**Study time:** 1 hour

**Concepts to remember**

Global, local and microscopic analysis, recrystallization, precipitation, deformation resistance, deformation speed, metal flow, temperature field

**Lecture**

The entire analysis is thus a means, which enables us to estimate how metal flow will be, if a required shape will be reached and what the mechanical parameters of a product will be.

For any forming problem we can design the general levels of a problem (viz. Table 4.1):

<table>
<thead>
<tr>
<th>Level</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. level – global analysis</td>
<td>Force</td>
</tr>
<tr>
<td></td>
<td>Work</td>
</tr>
<tr>
<td></td>
<td>Output</td>
</tr>
<tr>
<td></td>
<td>Temperature field</td>
</tr>
<tr>
<td>2. level – local analysis</td>
<td>Thermomechanical parameters</td>
</tr>
<tr>
<td></td>
<td>$(T, \varepsilon, \dot{\varepsilon}, \sigma)$</td>
</tr>
<tr>
<td>3. level – microscopic analysis</td>
<td>Grain size</td>
</tr>
</tbody>
</table>
1st level
The results of knowledge of the first level are essential for determining the basic technical parameters of forming machines and operations. The most important is the exact determination of the geometry of forming bodies before and after deformation, its mass, temperature field, required force and work from it and output. This enables either the design of classic (technologically manageable) production equipment, or it can lead to basic innovative changes.

2nd level
In determining basic thermomechanic quantities and their course in forming bodies today there is most often used the finite element method (FEM).

3rd level
Its composition is best illustrated in the development diagram Fig. 4.1., where there are captured basic links (full lines) and other interaction events are illustrated (by broken lines). The interconnectedness and complication of relations are greater here. Each individual problem and still other events (for example, precrystallization) individual attention has to be devoted to not losing the distinctly continuous whole. If we also consider precipitate events, the whole structure is even more complicated.

Material element

Dynamic changes
- hardening
- recovery
- static recovery
- static recrystallization
- grain growth
- homogenous
- non-homogenous
- recovery
- recrystallization
- deformation areas
- other phases
- post-recovery

Static changes
- static recovery
- static recrystallization
- grain growth
- homogenous
- non-homogenous
- recovery
- recrystallization
- deformation areas
- other phases
- post-recovery

Fig. 4.1. Multi-step simulation for solving forming parameters
Mutual links between the technological parameters of a process and by the main quantities of the first and second groups are illustrated in simplified form in Fig. 4.2. Variables from the third group are possible to connect with a given scheme through their effects on deformation resistance.

In total it is possible to say that for the successful use of simulation, leading to practical applications there are necessary at least these four assumptions:
1. The existence of a model scheme of metal flow and a model of heat transfer in differential equations and a suitable mathematical and computer program (on the basis of FEM).
2. Equations and constants for describing the curves of hardening, structural changes, temperature fields, etc.
3. Plastometric simulation, confirming the validity and functionality (at given limits) of theoretical relations and completing them with specific values.
4. Pilot simulation, for verifying a mathematical model and peripheral and initial conditions.

∑ Summary
In modelling we can research a problem on a global level (for which a functional analysis is enough for us) on a local level (where we most often meet with FEM) and on a microscopic level (when we simulate for example the behaviour of individual grains during recrystallization for example the MonteCarlo method).

For a correctly carried out analysis it is essential to have knowledge of the links between the variables in a system. You should be able to manage to find other links, which are not illustrated in Fig. 4.2.

Questions for Chapter 4
4.1. Can you explain the individual levels of the thermomechanical analysis of forming. Give examples.
4.2. How does a lubricant affect formability?
4.3. How does too much speed affect metal flow during forming?
4.4. Can you give an example on how it is possible to verify the peripheral conditions of mathematical analysis during semi-operational rolling?
4.5. What is a plastometer?

Questions to be solved for Chapter 4
4.1. Design a plastometric experiment for verifying a steel deformation resistance model.
Other sources


5. MATEMATICAL MODELS

Objective: After reading this chapter through you will know how:

- To distinguish models for describing metal flow
- To use models for describing heat transfers.
- To discuss possible errors while using incorrect models.
- To set up a complete model for describing microstructure development.

The content of Chapter 5

5.1. Metal flow model
5.2. Heat transfer model
5.3. Microstructure model

5.1. Metal flow model

Study time: 1 hour

Concepts to remember

Elastic-plastic formulation, Rigid-plastic formulation, Rigid-viscid-plastic formulation

Lecture

- FEM elastic-plastic formulation

For forming there is generally a rigid-plastic material model is preferred, where big plastic deformation dominates. Some factors, for example sheet forming, generally all problems solving residual stress, have to be simulated using an elastic-plastic material model.

a) Constitutive equation

In elastic areas the illustrated Hooke’s Law is valid:

\[ \delta_{ij} = \frac{1 + \nu}{E} \left( \sigma_{ij} - \frac{\nu}{1 + \nu} \cdot \delta_{ij} \cdot \sigma_{kk} \right). \]  \hspace{1cm} (5.1)

In plastic areas the Prandtl-Reuss (with HMH conditions of plasticity) theory including elastic and plastic areas:

\[ \delta_{ij} = \delta_{ij}^e + \delta_{ij}^p, \]  \hspace{1cm} (5.1)

\[ \sigma_{ij} = \lambda \cdot \sigma_{ij}^p, \]  \hspace{1cm} (5.2)
\[ \sigma'_{ij} - \sigma'_{jj} = \frac{2}{3} \bar{\sigma}^2 \tag{5.3} \]
\[
d\sigma_j = 2 \cdot G \left[ \epsilon_{ij} + \delta_j \cdot \frac{\nu}{1 - 2 \cdot \kappa} \cdot d\epsilon_{kk} - \sigma'_{ij} \cdot \frac{\epsilon_{kk}}{S} \right] \tag{5.4} \]

where \( \sigma_j \) is the stress deviator,
\( \kappa \) is Poisson’s constant,
\( G \) is the elasticity module in slippage,
\( \delta_j \) is Kronecker’s delta (for \( i = j \) is \( \delta_j = 1 \); for \( i \neq j \) is \( \delta_j = 0 \)),
\( S \) is a material constant,
\( \bar{\sigma} \) is stress intensity.

b) Formulation
1. It is used with Lagrange’s co-ordinate system (instant deformation is in relation to the original non-deformed configuration).
2. The problem is solved in the area of stress or in deformation.
3. The unknown quantity is shift.
4. The basic functional is founded on the variation principle or on the principle of virtual work.
5. There is used for big deformation an incremental solution.

c) Advantages
1. It takes into consideration the transfer from an elastic to a plastic state (for example. Poisson’s constant \( \kappa = f(\epsilon) \)).
2. In the solution there is included elastic and plastic areas.
3. It is considered with geometric nestability and non-homogeneity.
4. It is possible to affect residual stress, cushioning and friction.

d) Disadvantages
2. High demands on computing time, especially if there is used flow theory.
3. Numerical errors can accumulate.

- **FEM rigid-plastic formulation**

  a) Constitutive equation

  Levy-Mises material law:

  \[ \sigma'_{ij} = \frac{2}{3} \bar{\sigma} \cdot \dot{\varepsilon} \] \[ \text{[Pa]} \tag{5.5} \]

  where \( \bar{\sigma} < R_\epsilon \) for rigid material,
  \( \bar{\sigma} = R_\epsilon \) for plastic material,
  \( R_\epsilon \) yield limit,
  \( \dot{\varepsilon} \) is the intensity of the deformation speed.

  b) Formulation

  1. There is used Euler’s co-ordinate system (deformation is related to a momentary value).
2. An unknown is the speed field.
3. Problems are connected to the requirements of fulfilling the non-compressability conditions in functional
   \[ \Pi = \int_{\Omega} \bar{\sigma} \cdot \bar{\varepsilon} - \int_{\partial \Omega} T_i \cdot u_i \cdot dS, \]  
   it is solved using a penalized function or Lagrange’s multiplier.
4. The analysis of the non-stationary forming process is carried out incrementally by means of a great number of small consistent (stationary) deformation steps.
   
   c) Advantages
   1. In every iteration step a linearized relation is given between stress and speed deformation.
   2. For non-stationary problems there is used a quasi-stationary solution.
   3. Short computing time.
   
   d) Disadvantages
   1. It is not possible to include elastic loading.
   2. It is not possible to calculate geometric non-linearity and non-homogeneity.

- **FEM rigid-viscid-plastic formulation**
  a) Constitutive equation
   The material is considered as a „non-newtonian“ liquid, for which there is adapted HMH plasticity conditions.
  
  b) Solution formulation
   The same as a rigid-plastic formulation.
  
  c) Advantages
   1. It is possible to simulate hot forming.
   2. Every iteration step has the physical sense of a non-stationary process.
  
  d) Disadvantages
   1. The elastic deformation is neglected.
   2. The system of non-linear equations is sensitive to co-efficient viscosity.

- **Peripheral conditions**
  The peripheral stress condition for \( \partial \Omega_p \) can be either zero or in the best case described by a uniform hydrostatic pressure. However on the contact surface between the tool and semiproduct the peripheral condition is usually mixed. Besides this, neither speed nor force (calculated in size and direction), cannot be completely required on the contact surface, because friction stress acts in an opposite direction than the mutual speed between a semiproduct and a tool, and this speed is known a priori.
5.2. Heat transfer model

Study time: 1 hour

Concepts to remember

- Biot’s criterion, peripheral conditions, heat transfer co-efficient

Lecture

- Introduction

  Relatively little knowledge about the heat course in metal during rolling is connected to the known complexity of this process, for the temperature change of rolled metal is affected by heat removal, especially convection and heat emission, and heat transfer during metal deformation. A further solution complexity in the heat field changes of rolled metal consists in the fact that it deals with, in its entire volume, the non-stationary transfer of heat, thus with inconsistent time-thermal events (Fig. 5.1).
Fig 5.1 Average bar temperature at the 1st. rolling stand. (HCC, ArcelorMittal Ostrava)
The temperature drop in rolling profiled steel is rather different than during the rolling of blocks and slabs, but the complexity of the profile shape introduces a whole series of specialities into the temperature distribution along its cross-section. The basic difference is in the fact that during the passage of rolling stock through a groove during plastic deformation the metal flow becomes more complicated than during rolling on smooth rolls, which considerably complicates determining the initial solution conditions during each passage through the groove. Another complication forms in that the rolled profile rolling stock does not uniformly cool, they form thermal points and thinner profile parts cool faster, which makes the thermal field along its cross-section non-uniform. A typical example is rolling a R65 rail into a foot, that is a slit, or groove, see Fig. 5.2., when during separating a future

**Fig. 5.2.** Surface temperatures when rolling rails R 65 at a 5th groove: a) a diagram of measured temperatures using a pyrometer, b) surface photography.
head and foot the crest of a groove penetrates deeply into the interior of rolling stock and thus exposes the interior metal layer with a much higher temperature.

The rolling stock temperature change in the course of the rolling process is possible to simply state as a change in the average temperature of rolling stock, or more precisely as a change of the thermal field along a rolling stock cross-section along the length of the rolling line. For calculation calibrations it is now considered with the average rolling stock temperature, which does not capture the temperature conditions for complicated shape ranges. Determining the thermal field along a cross-section is more difficult in regards to defining peripheral conditions.

- **Mathematical modelling of temperature changes when rolling rolled sections**

  In dealing with the thermal field changes of rolled material it is possible to generally use an analytical or numeric solution method. The analytic solution of Fourier’s differential equation describing non-stationary heat conductivity is possible to carry out using different methods.

  Fourier’s differential equation for a body described in a Cartesian co-ordinate system has the following form:

  $$\frac{\partial t}{\partial \tau} = a \cdot \left( \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right) \quad \text{[K.s}^{-1}\text{]}, \quad (5.7.)$$

  where $a$ is the temperature conductivity co-efficient $[\text{m}^2.s^{-1}]$, $t$ is temperature $[\text{K}]$, $\tau$ is time $[\text{s}]$, $x,y,z$ are co-ordinates $[\text{m}]$.

- **Unambiguity conditions**

  Correctly determining the kinematics of temperature field changes is dependent on unambiguity conditions:

  - **physical conditions** – heat conductivity co-efficient, density, specific thermal capacity, etc.;
  - **geometric conditions** – rail shape and dimensions;
  - **peripheral conditions (initial + surface)** – characterizing initial temperature distribution in a body and mutually acting between a body and its surroundings.

  **Initial conditions** define temperature distribution in rolling stock with an initial solution state. During solution there is used especially two kinds of initial conditions:
  1. Temperature is constant at the initial time of solution in all rolling stock points.

$$t = t_0 = \text{konst} \quad \text{const}$$,

(5.8)

2. Temperature distribution at the initial time of solution is described by a known functional dependence (Fig. 5.3.).

$$t(x,y,\tau_{poč}) = \text{konst} \quad [°C]$$,

(5.9)
The temperature distribution for complicated rolled sections is affected by rolling conditions, which for example is not possible to state using the relation of the lengths of arch passes to the initial material thickness. The course of thermal increase under the influence of plastic deformation is then possible to state as

\[
\Delta t_{\text{def}} = f_1(y) \quad [\text{K}], \\
\Delta t_{\text{def}} = f_2(y) \quad [\text{K}], \\
\Delta t_{\text{def}} = f_3(y) \quad [\text{K}],
\]

where \( f_1 \) is function \( \Delta t_{\text{def}} \) according to Fig. 5.5a, \\
\( f_2 \) is function \( \Delta t_{\text{def}} \) according to Fig. 5.5b, \\
\( f_3 \) is function \( \Delta t_{\text{def}} \) according to Fig. 5.5c,

\[
l_d = \sqrt{R \cdot \Delta h} \quad \text{length of deformation zone [m]},
\]

\( R \) – roll radius [m].
\( \Delta h \) – absolute height deformation [m].

\[ \begin{align*}
\Delta t_{\text{def}} & \quad \Delta t_{\text{def}} \\
\end{align*} \]

**Fig. 5.5.** Course of temperature increment for rolling conditions.

In addition to the formation of deformation heat in rolling, there exists a whole series of other agents, which affect temperature distribution along a rail cross-section, as for example heat formation due to friction, rolling stock contact with conveyors and to a large extent rolling stock contact with rolls, when it leads to intensive heat transfer, which is affected by a whole series of difficult to define parameters (the presence of iron scales, roll cooling using water, etc.).

By heating up a research sample in an oven it is thus not possible to reach the same distribution of temperature, as for a rolled material, and this is especially for a complicated profile, such as rails. The difference in the initial conditions of a mathematical model and real rails has shown the wrong solution in the course of calculations. This error attains its greatest value in the initial cooling phase and the temperature consequently decreases.

Surface conditions characterize the thermal relation on rolling stock surfaces. It is generally possible to define four kinds of surface conditions:

**Condition of the 1st type (Dirichlet)**
At any time moment the temperature distribution on the surface is known:

\[ t_p = f(t) \quad , \quad (5.13) \]

where \( t_p \) is surface temperature [°K],
\( t \) is time [s].

**Condition of the 2nd type (Neuman)**
At any time step the density of thermal flow through the surface is known

\[ -\lambda \frac{\partial t}{\partial x} |_{x=0} = q = f(z(t)) \quad , \quad (5.14) \]

where \( \lambda \) is the thermal conductivity co-efficient [W.m\(^{-1}\).K\(^{-1}\)],
\( q \) is the density of heat flow [W.m\(^{-2}\)].

**Condition of the 3rd type (Fourier)**
At every time moment the temperature of the surrounding environment is known and it is given by the mathematical equation for heat transfer between an environment and body surface

\[ -\lambda \frac{\partial t}{\partial x} |_{x=0} = \alpha \cdot (t_p - t_0) \quad , \quad (5.15) \]
where $\alpha$ is the co-efficient of heat transfer [W.m$^{-2}$.K$^{-1}$].

**Condition of the 4th type**

A body surface displays the thermal effect of another body with which it is in direct contact. Contact on the surface of both bodies is completely perfect, so that the temperature of contacting body points is the same. This condition thus states the equality of thermal flow on the edges of both bodies

$$-\lambda \frac{\partial t_1}{\partial x_1} \bigg|_{x=0} = -\lambda \frac{\partial t_2}{\partial x_2} \bigg|_{x=0}$$

(5.16)

Surface conditions allow for heat removal during rail cooling after rolling. The use of the surface condition of the 2nd type is difficult in practical measuring. From a lack of knowledge of the surface conditions of the 1st type it is more suitable to use the surface condition of the 3rd type, which requires a knowledge of the total co-efficient of heat transfer by convection and heat emission $\alpha_C$. It is possible to state value $t_p$ by extrapolation of the temperature gradient values of the experimental measuring of the internal temperatures of the rail sample.
Open cooling in the air

Recently most standard rails are cooled by this method. When temperature is shared there is shown steel surface radiation and uncontrolled convection. This method is the simplest, but a low intensity of cooling is shown. For determining co-efficient \( \alpha \) it is possible to use these relations:

\[
\alpha_c = \alpha_k + \alpha_s \quad \text{[W.m}^2\text{.K}^{-1}] ,
\]

where \( \alpha_k \) is the co-efficient of convention thermal conductivity [W.m\(^2\).K\(^{-1}\)],

\( \alpha_s \) is the co-efficient of heat emission thermal conductivity [W.m\(^2\).K\(^{-1}\)].

Determining the co-efficient of convection heat transfer:

\[
\alpha_k = \frac{Nu \cdot \lambda}{l} \quad \text{[W.m}^2\text{.K}^{-1}] ,
\]

where \( Nu = c \cdot (Gr \cdot Pr)^n \) is Nusselt’s criterion (sharing heat by convection) [ - ],

\( Gr = \frac{g \cdot \gamma \cdot \Delta T \cdot l^3}{\nu^2} \) is Grasshoff’s cr. (nat. convection of viscal liquid) [ - ],

\( Pr = \frac{\nu}{a} \) is Prandtl’s criterion (sharing heat in liquids) [ - ],

\( \lambda \) is the co-efficient of thermal conductivity [W.m\(^{-1}\).K\(^{-1}\)],

\( l \) is the characteristic dimension [m],

\( g \) is gravitation acceleration [m.s\(^{-2}\)],

\( \gamma \) is thermal volumic expandibility [K\(^{-1}\)],

\( \Delta T \) is the temperature difference [K],

\( \nu \) is the kinematic viscosity of liquid [m\(^2\).s\(^{-1}\)],

\( a \) is the co-efficient of thermal conductivity [m\(^2\).s\(^{-1}\)].

<table>
<thead>
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<th>Validity range</th>
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<th>( n )</th>
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<td>1,180</td>
<td>0,125</td>
</tr>
<tr>
<td>( 5 \cdot 10^2 &lt; Gr \cdot Pr &lt; 2 \cdot 10^7 )</td>
<td>0,540</td>
<td>0,250</td>
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<tr>
<td>( 2 \cdot 10^7 &lt; Gr \cdot Pr &lt; 1 \cdot 10^{13} )</td>
<td>0,135</td>
<td>0,330</td>
</tr>
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</table>

Determining the co-efficient of heat transfer radiation:

\[
\alpha_s = C_0 \cdot \varepsilon_n \cdot \left[ \left( \frac{T_p}{100} \right)^4 - \left( \frac{T_{ok}}{100} \right)^4 \right] \quad \text{[W.m}^2\text{.K}^{-1}] ,
\]

where \( \varepsilon_n \) the co-efficient of relative absorption, which is determined from general relation:

\[
\varepsilon_n = \frac{1}{\left( \frac{1}{\varepsilon_1} - 1 \right) \phi_{12} + \left( \frac{1}{\varepsilon_2} - 1 \right) \phi_{21} + 1} \quad \text{[ - ]} ,
\]
where $\varepsilon_1$, $\varepsilon_2$ is the co-efficient of relative absorption for surface 1; 2, 
$\varphi_{12}$, $\varphi_{21}$ are angular co-efficients of heat transfer emitted from surface 1 to surface 2 and from surface 2 to surface 1.

Surface 1 is made up of the surface of rails, head, foot, side walls. Surface 2 is made up of the surrounding area or cooling bed.

For the upper head area and the lower foot area of the lying rails there is angular co-efficient $\varphi_{12} = 1$. For the vertical rail walls there is angular co-efficient

$$\varphi_{12} = \frac{S_3}{S_1} \quad [-] \quad (5.24)$$

where $S_3$ is the fictive area of a given rail length and the distance between the foot end and the lower head end in meters,

$S_2$ is the area of interior rails wall in meters.

For heat emission of the lower rail walls on the cooling bed it is possible to assume the heat emission of two parallel areas, then $\varphi_{12} = 1$ and $\varphi_{21} = 1$.

In Fig. 5.5, there is shown the co-efficient dependence of heat transfer on temperature and the emissivity of a cooled surface. The graph was calculated with the program TTSteel 2.0 on the basis of equations (5.22) and (5.24).

### Fig. 5.5. Dependence $\alpha$ on temperature and the emissivity of a cooling surface at an air temperature of 20 °C.

- **Accelerated cooling using water**

Water is the most intensive hardening medium. The advantage of it is its high cooling efficiency in the area of perlitic modifications. Cooling is founded especially on water evaporation. In high temperatures of steel the forming bubbles are connected in a continuous vapour membrane and membrane boiling (at $T_p \approx 250 ^\circ C$) begins. There is created a thin layer of water vapour, that is a vapour cushion. This layer acts as an insulator and cooling intensity drops. Radiation throughout the layers and conductivity share in heat distribution. The
Matematické modely

distribution of the vapour cushion helps water circulation, as does adding suitable ingredients into the bath (hydroxides, salts), or it is possible to use a water spray for cooling (very intensive heat removal). To determine the co-efficient size there can be for example from a thermal balance:

\[
\Delta Q_{chł} = \alpha \cdot (T_p - T_v) \cdot S \cdot \tau_{chł} \quad [J],
\]

where \( \alpha \) is the co-efficient of heat transfer \([\text{W.m}^{-2}\cdot\text{K}^{-1}]\),
\( T_p \) is the surface temperature before cooling \([\text{°C}]\),
\( T_v \) is the water temperature \([\text{°C}]\),
\( S \) is surface \([\text{m}^2]\),
\( \tau_{chł} \) is the cooling period \([\text{s}]\).

\[
\Delta Q_{chł} = m \cdot c \cdot \Delta T \quad [J],
\]

where \( m \) is mass \([\text{kg}]\),
\( c \) is the specific temperature of rolling stock \([\text{J.kg}^{-1}\cdot\text{K}^{-1}]\),
\( \Delta T \) is the surface temperature drop \([\text{°C}]\).

In Fig. 5.6. the co-efficient dependence is given for heat transfer at the temperature of a cooled surface for calm water at temperatures 30 and 70 °C (source: database program TTSteel 2.0).

In Fig. 5.6. Dependence \( \alpha \) at a temperature for water at 30 and 70 °C.

- Accelerated cooling by air

There is used an air stream from a blower or also jets with compressed air. The intensity is less than that for water, but there is the risk of the formation of a vapour cushion and the risk of supercooling. Most heat removal is by convection (by forced flowing), with co-
efficient $\alpha \approx 150 - 800 \text{ W.m}^{-2}.\text{K}^{-1}$. For closer determination of co-efficient $\alpha$ it is possible to use these relations:

Bypassing rolls:

\[ Nu = C \cdot \text{Re}^m \cdot \text{Pr}^n \cdot \chi \cdot \varepsilon_{\psi} \quad [-] \quad \text{(5.27)} \]

Bypassing prisms:

\[ Nu = C \cdot \text{Re}^m \cdot \text{Pr}^{0,33} \quad [-] \quad \text{(5.28)} \]

where $Re$ is Reynolds’ criterion, $Pr$ is Prandtl’s criterion, $\chi$ is the thermal co-efficient, for gases $\chi = 1$, $\varepsilon_{\psi}$ is the co-efficient including the angle of flow impact.

**Table 5.2.** Constant values in equations (5.27), (5.28)

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<tr>
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<th>m</th>
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<td>0,5</td>
<td>0,38</td>
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<td>$1 \cdot 10^3 &lt; \text{Re} &lt; 2 \cdot 10^5$</td>
<td>0,250</td>
<td>0,6</td>
<td>0,38</td>
</tr>
<tr>
<td>$2 \cdot 10^5 &lt; \text{Re} &lt; 2 \cdot 10^6$</td>
<td>0,023</td>
<td>0,8</td>
<td>0,37</td>
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</tbody>
</table>

In Fig. 5.7, there is shown the the co-efficient dependence of heat transfer at the temperature of the cooled surface and the speed of air flow (emissivity is 0,8, which corresponds to an iron scale steel surface above 500 °C). The graph was calculated using program TTSteel 2.0 according to equation (5.28).

Heat transfer co-efficient | Temperature | Flow speed

![Graph showing the dependence of heat transfer co-efficient on temperature and flow speed](image)

**Fig. 5.7.** Dependence $\alpha$ on the temperature of a cooled surface with a speed of air flow for air temperature 20 °C and emissivity 0,8.

- Rolling stock contact with rolls
Summary
Not only in hot forming, but also in cold rolling heat conductivity plays a role. In addition to the correct entering of material thermal properties it is most important to determine the heat transfer co-efficient. While the calculation of body cooling into the surrounding air is relatively easy and precise, if another cooling medium is used (water, oil) and or it deals with contact with a tool, the situation is far more complicated.

5.3. Microstructural model

Study time: 1.5 hours

Concepts to remember
Dynamic, metadynamic and static recrystallization, precipitation, recovery

Lecture

Summary
You have become acquainted with a complex model for describing the development of a microstructure. It is not necessary to remember individual equations even if you possibly know them from other subjects. What is important is the common logical connections of the individual submodels.

Questions for Chapter 5

5.1. What is the importance of number -0.693 in a AJMK equation?
5.2. On what does time depend for recovering half the structure by static recrystallization?
5.3. What postdynamic recovery processes do you know?
5.4. What is the condition for forming metadynamic recrystallization?
5.5. On what does the original grain size depend?
5.6. What affects the residual deformation of the size of a ferritic grain?
5.7. What is precipitation?
5.8. Can you define the interlamear perlite distance?
5.9. On what does the strength of ferrite-peritic steel depend?
### Problems to solve in Chapter 5

5.1. In the Excel program design a table, which for the entered thermomechanic parameters will calculate the change of grain size in rolling on a continuous rolling belts with 7 pass reductions.

### Other sources


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<table>
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<th>Problems to solve in Chapter 5</th>
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<tr>
<td>5.1. In the Excel program design a table, which for the entered thermomechanic parameters will calculate the change of grain size in rolling on a continuous rolling belts with 7 pass reductions.</td>
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### Other sources


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### První Tabulka

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<td>Process analysis and optimization</td>
<td>Determining the course of geometric changes and performed process work</td>
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<td>Loading, energy, tension deformation, temperature metal flow (geometric changes)</td>
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<td>Forming limits</td>
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<td>Dimensional tolerance surface properties, etc.</td>
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